Direction of Arrival Estimation using a Sparse Linear Antenna Array

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***Abstract*—This paper presents the use of Inverse Free Krylov Subspace Algorithm (IFKSA) with Estimation of Signal Parameter via Rotational Invariance Technique (ESPRIT), for Direction-of-Arrival (DOA) estimation using a Sparse Linear Array (SLA) of antenna elements. The use of SLA reduces the hardware requirements and the production cost. The SLA generates the sparse data, the pure signal subspace is first decomposed from the received corrupted signal space using IFKSA and later, ESPRIT is used to estimate the DOAs. Computer simulations are used to evaluate the performance of the proposed algorithm.**

***Index Terms*—Krylov subspace, ESPRIT, DOA, Sparse linear array**

I. Introduction

Estimating the Direction-of-Arrival (DOA) of the far-field narrowband signals has been of research interest in array signal processing literature [1]. The applications of the DOA estimation is found in RADAR, SONAR and other Wireless communication systems in the areas like localization, tracking, navigation and surveillance. There are number of DOA estimation algorithms like Multiple Signal Classification (MUSIC) [2], Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) [3] and their derivatives [4][5][6]. These algorithms rely on the uniform structure of the data, which is obtained from the underlying Uniform Linear Array (ULA) of antennas (sensors). ULA consists of a number of elements with inter-element spacing equal to half the wavelength of the impinging signal source. The size of the ULA increases the hardware requirement. The same performance with slight modification in the Mean Square Error (MSE) can also be obtained by using the Sparse Linear Array (SLA) of antennas or the Nonuniform Linear Array (NULA) of antennas[7][8]. In the SLAs, the inter-element spacing need not be maintained and the reduced number of elements further reduces the hardware requirement and the production cost. Moreover, the SLA gives the same performance as the ULA, with reduced number of elements. Furthermore, failure of a few elements at random locations will also result in SLA.

Among the various high resolution methods for DOA estimation, subspace based methods are most popular and powerful. The popularity is due to its strong mathematical model to illustrate the underlying data model and it can with stand the perturbations in the data. Subspace methods for DOA estimation searches for the steering vector associated with the directions of the signals of interest that are orthogonal to the noise subspace and are contained in the signal subspace. Once the signal subspace is extracted the DOAs are estimated. The decomposition is performed using the Eigen Value Decomposition (EVD) of the estimated received signal correlation matrix. The MUSIC and ESPRIT are the popular subspace based DOA estimation algorithms. MUSIC algorithm is a spectral search algorithm and requires the knowledge of the array manifold for stringent array calibration requirement. This is normally an expensive and time consuming task. Furthermore, the spectral based methods require exhaustive search through the steering vector to find the location of the power spectral peaks and estimate the DOAs. ESPRIT overcomes these problems by exploiting the shift invariance property of the array. The algorithm reduces computational and the storage requirement. Unlike MUSIC, ESPRIT does not require the knowledge of the array manifold for stringent array calibration. There are number of variants and modification of ESPRIT algorithm. ESPRIT algorithm is also extended for sparse linear antenna arrays or non-linear antenna arrays. For nonlinear arrays the aperture extension and disambiguation is achieved by configuring the array geometry as dual size spatial invariance array geometry [9] or by representing the array as Virtual ULA, and using the Expectation-Maximization algorithm [10]. However, the subspace algorithms are heavily dependent on the structure of the correlation matrix and are unsuitable to handle sensor failures.

In this paper, we extend the conventional ESPRIT to estimate the DOAs using the SLA. The basic idea of any subspace based approach is; decomposing the received corrupted signal space in to pure signal subspace and noise subspace. This is generally achieved using the Eigenvalue decomposition. However, the sparse data output from the SLA cannot be directly decomposed, because it deteriorates the performance of the subspace based approaches. We here, use the Krylov subspace based methods [11] to decompose the received corrupted sparse signal space and later use ESPRIT to estimate the DOAs. The resulting algorithm is called as IFKSA-ESPRIT DOA estimation algorithm. There are number of Krylov subspace based techniques. The iterative methods like Lanczos algorithm, Arnoldi algorithm and Jacobi-Davidson algorithm are very

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popular among them [12]. The methods are very sensitive to perturbations and are shift-and-invert procedures leading to increased computational complexity. Inverse Free Krylov Subspace algorithm proposed by Golub and Ye [13], is an attractive eigenvalue computation algorithm. IFKSA iteratively improves the approximate eigen pair, using either Lanczos or the Arnoldi iterations at each step through Rayleigh-Ritz projection procedure [11]. The algorithm is a very attractive due to the following reasons; first, the technique can be used to find any number of smallest eigenvalues (Largest can also be calculated), and second, the algorithm is less sensitive to perturbations. The performance of the algorithm is evaluated for various sparse antenna array geometries.

The rest of the paper is organized as follows. In the following section the signal model is discussed. In section III, the proposed DOA estimation algorithm is presented. Section IV, discusses the simulation results and finally, conclusions are drawn in section V.

II. Signal Model

In this section signal model is discussed, beginning with the ULA and later we discuss for the SLA. The DOA estimation problem is to estimate the directions of plane wave incident on the antenna array. The problem can be looked as parameter estimation. We here mainly introduce the model of a DOA estimator. Consider an *M* - element uniformly spaced linear array. The array elements are equally spaced by a distance *d*, and a plane wave arrives at the array from a direction θ off the array broadside. The angle θ is called the direction-of-arrival (DOA) or angle-of-arrival (AOA) of the received signal, and is measured clockwise from the broadside of the array.

Let *N* narrowband signals $s_1(t), s_2(t), \cdots, s_N(t)$ all centered around a known frequency, impinging on the array with a DOA $\theta_i,\ i = 1,2,\cdots N$. The received signal at the array is a superposition of all the impinging signal and noise. Therefore, the input data vector may be expressed as

$$\mathbf{x}(t) = \sum_{i=n}^{N} \mathbf{a}(\theta_i) s_i(t) + \mathbf{n}(t) \tag{1}$$

Where,

$$\mathbf{a}(\theta_i) = \left[1, e^{-j\frac{2\pi}{\lambda} d \sin\theta}, \cdots, e^{-j\frac{2\pi}{\lambda} d(M-1)\sin\theta}\right]^T \tag{2}$$

$\mathbf{a}(\theta_i)$, is the steering vector of the array in the direction θ_i. To further simplify the notation, we write (2) as

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{s} + \mathbf{W} \tag{3}$$

Where , X is the array output matrix of size $M \times K$, A is the complete steering matrix of size $M \times N$ function of the DOA vector $\boldsymbol{\theta}$, is signal vector of size and is the noise vector of size M X K . Here, K is the number of snapshots. Here, (3) represents the most commonly used narrowband input data model.

We are primarily interested in a SLA, where, some elements at some locations are missing. An example of a SLA is shown in Fig. 1, where elements at locations 10, 7 and 3 are missing,

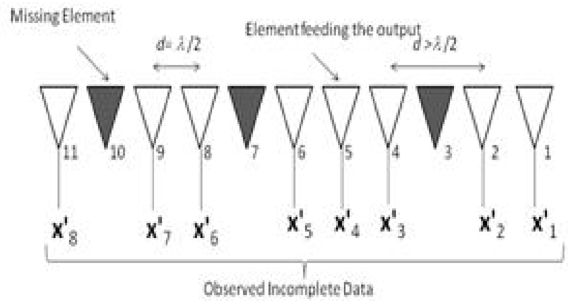

Figure. 1. Example of a SLA. Elements at Location 3, 7 and 10 are Missing.

and the remaining elements are feeding the output.The $\acute{\mathbf{X}}$, output of a SLA will be represented as a sparse data given as

$$\acute{\mathbf{X}} = \acute{\mathbf{A}}(\boldsymbol{\theta})\mathbf{s} + \acute{\mathbf{W}} \tag{4}$$

Where, $\acute{\mathbf{X}}_{\acute{M}\times K}$ is the sparse data generated by SLA of $\acute{M}$ elements, $\acute{\mathbf{A}}(\boldsymbol{\theta})_{\acute{M}\times N}$ is the steering matrix of the SLA, and $\acute{\mathbf{W}}_{\acute{M}\times K}$ is the noise.

The estimated correlation matrix of (4) of the sparse represented data is written as

$$\acute{\mathbf{R}} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{x}_k{}'\mathbf{x}_k{}'^H \tag{5}$$

Therefore, the problem is to decompose the correlation matrix in (5) to obtain the signal subspace, which is further processed using the ESPRIT, to estimate the DOAs.

III. IFKSA-ESPRIT Algorithm

The IFKSA-ESPRIT algorithm for estimating the DOAs from a faulty ULA is given in Table I. It consists of two steps. Step 1, is decomposing the correlation matrix given in (5) using the IFKSA technique. In the second step, the conventional ESPRIT algorithm is applied to estimate the DOAs.The IFKSA algorithm for finding the approximate *N* largest eigenvalues and the eigenvectors $(\tilde{\lambda}, \mathbf{v})$ of the correlation matrix $\acute{\mathbf{R}}$ given in (5) is described as follows. The IFKSA algorithm computes the smallest eigenvalues of $\acute{\mathbf{R}}$. To compute the largest eigenvalues replace by $\acute{\mathbf{R}}$ and - R compute its smallest eigenvalues and reverse the sign to obtain the largest eigenvalues of [13].

Given an initial approximation, the goal is to improve the initial approximation through the Rayleigh-Ritz orthogonal projection on a certain subspace, i.e., by minimizing the Rayleigh quotient on that subspace.

$$\rho(\mathbf{v}_k) = \frac{\mathbf{v}_k{}^T \acute{\mathbf{R}} \mathbf{v}_k}{\mathbf{v}_k{}^T \mathbf{v}_k} \tag{6}$$

The IFKSA is a technique for computing the new approximate eigenvectors $\mathbf{v}_{k+1}$.The technique starts with an approximate eigenvectors $\mathbf{v}_k$ and constructs a new

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TABLE I. INVERSE FREE PRECONDITIONED KRYLOV SUBSPACE - ESPRIT ALGORITHM (IFKS-ESPRIT)

Input R:
Output: The estimated DOAs

Step 1: Compute the Eigen decomposition of the array covariance matrix using the Inverse Free preconditioned Krylov subspace (IFKS) algorithm

IFKS Algorithm ( ):

Step a: Set $n \geq 1$, $\acute{R}$ and an initial approximation v_0 with $||v_0|| = 1$

Step b: For $k = 0, 1, \cdots, untill\ Convergence$
Construct a basis $U_m = [u_0, u_1, \cdots, u_n]$ for the subspace
$\mathcal{K}^m = span\left(v_k, (\acute{R} - \rho_k)v_k, (\acute{R} - \rho_k)^2 v_k, \cdots, (\acute{R} - \rho_k)^{n-1} u_k\right)$

Step c: Form $\acute{R}_m = U_m{}^T(\acute{R} - \rho_k)U_m$

Step d: Compute the (β_i, w_i) eigenpairs of $\acute{R}_m$, and select the desired ones.
$\rho_{k+1} = \rho_k + \beta_i$ $\quad v_{k+1} = U_m w_i$

End;

Step 2: Estimate the DOA using ESPRIT

ESPRIT ( ):

Step a: Form the signal subspace V_s using the eigenvectors obtained from step 1

Step b: Form the matrices V_{s_0} and V_{s_1}

Step c: Estimate the DOA solving $[V_{s_0}, V_{s_1}]$.

End;

approximate by Rayleigh-Ritz projection of on to the Krylov subspace.

$$\mathcal{K}^m = span\left(v_k, (\acute{R} - \rho_k)v_k, (\acute{R} - \rho_k)^2 v_k, \cdots, (\acute{R} - \rho_k)^n u_k\right) \quad (7)$$

where n is the fixed parameter, generally taken between $1 \leq n \leq 100$ [15].

The projection is carried out by constructing a matrix of basis vector U_n for $\mathcal{K}^n$, then forming and solving the projection problem for . The process is repeated for the required number of eigenpairs. This iteration forms the outer iterations. In the outer iterations, using U_n,

$$\Phi_m = U_m{}^T(\Phi - \rho_k)U_m \quad (8)$$

is formed. And the eigenpairs (β_i, w_i) for $\acute{R}$ is computed. Then the new approximate of the eigenvector is

$$v_{k+1} = U_m w_i \quad (9)$$

and correspondingly, the Rayleigh-Ritz quotient

$$\rho_{k+1} = \rho_k + \beta_i \quad (10)$$

is a new approximate eigenvalue. Now, to construct the orthonormal basis vector U_n, he Lanczos algorithm [11] is used as inner iteration, the algorithm is given in Table 2. The Lanczos algorithm starts with an initial vector v_0 One matrix vector multiplication $g = C_k u_i$ performed in every iteration, where is the residual. To make sure that u_{j+1} is orthogonal to u_{j-1} and u_j and extra orthogonalisation against these two vectors are done in further steps shown in Table II.

The steps of the IFKSA are repeated until convergence. Once the signal subspace is computed from the N eigenvectors v_k, the DOAs are estimated using

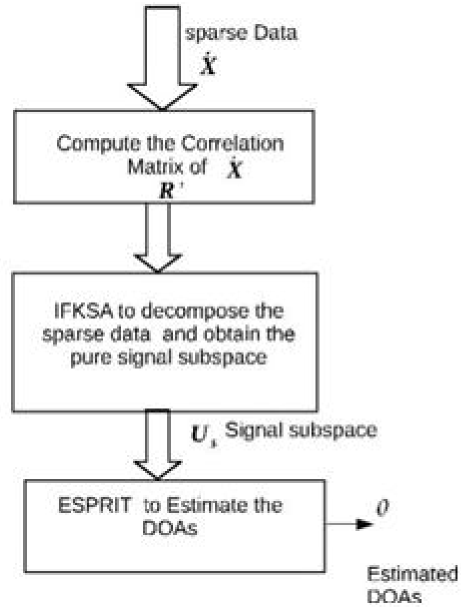

Figure2: Implementation of the IFKSA-ESPRIT Algorithm.

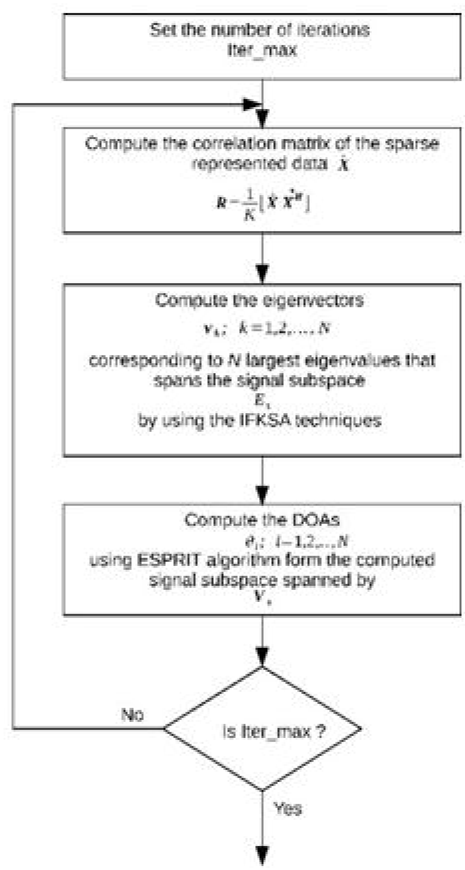

Figure 3: Flowchart of the Proposed Algorithm

conventional ESPRIT method. The implementation of the proposed algorithm is illustrated in Fig. 2. The flowchart of the proposed IFKSA-ESPRIT algorithm is shown in

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Fig. 3. The flowchart starts with initializing the maximum number of iterations, for every iteration the estimated correlation matrix of the sparse represented data for K number of snapshots is computed. From the computed estimated correlation matrix, the large N eigenvectors are calculated from the IFKSA algorithm. Using these N

TABLE II. LANCZOS PROCESS FOR CONSTRUCTING THE ORTHONORMAL BASIS

Input $\mathbf{C}_k = (\mathbf{R} - \rho_k)$, an approximate eigenvector: $\mathbf{v}_k$ Ouput $\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_n]$
$\mathbf{u}_0 = \mathbf{v}_k / \lVert\mathbf{v}_k\rVert_2$; $\gamma_0 = 0$; $\mathbf{u}_{-1} = 0$;   for $i = 0 : (n-1)$ $\mathbf{g} = \mathbf{C}_k \mathbf{u}_i$ $\mathbf{g} = \mathbf{g} - \gamma_i \mathbf{u}_{i-1}$ $\alpha_i = \mathbf{u}_i^T \mathbf{g}$ $\mathbf{g} = \mathbf{g} - \alpha_i \mathbf{u}_i$ $\gamma_i = \mathbf{g} / \lVert\mathbf{g}\rVert_2$ $\mathbf{u}_i = \mathbf{u}_i / \gamma_i$ End;

TABLE. III. SUMMARY OF THE IFKSA-ESPRIT ALGORITHM WITH COMPUTATIONAL COMPLEXITY

Algorithm	Computational Complexity
$\hat{\mathbf{R}} = \frac{1}{K}[\hat{\mathbf{X}}\hat{\mathbf{X}}^H]$	KM^2
Calculate [eigenvalues, eigenvectors] using IFKSA method	$\mathcal{O}(M^2)(n+1)$
Select N largest eigen pair to obtain the signal subspace $\mathbf{V}_s$	
Form $\mathbf{V}_{s0}$ by deleting the first column of $\mathbf{V}_s$	
Form $\mathbf{V}_{s1}$ by deleting the last column of $\mathbf{V}_s$	
Compute Eigenvalues $= \mathrm{eig}(\mathbf{V}_{s0}, \mathrm{pinv}(\mathbf{V}_{s1}))$	$\mathcal{O}(N^2)$
Calculate DOAs	

larger eigenvectors the conventional ESPRIT is executed to estimate the required DOAs. The results are computed and stored for each iteration for evaluation, until the maximum number of iterations. The summary of the algorithm with the computational complexity is presented in Table III.

IV SIMULATION RESULTS

In this section, we examine the performance of the proposed IFKSA-ESPRIT algorithm. The signal model consists of two equal magnitude complex analytical signal impinging on a 20 element ULA from direction $[10^\circ, 45$ The interelement spacing in ULA is equal to half the wavelength of the signal. The received signal is noncoherent and corrupted due to Additive White Gaussian Noise (AWGN). All examples assume 200 snapshots and 100 simulation trails are conducted. The simulations are performed using Matlab 7, on a Intel core 2 Duo processor installed with Windows Xp SP2 operating system.

The signal is assumed as a complex exponential sequences, given by

$$s(t) = \exp\left(j\phi(t)\right), \quad t = 1,2, \cdots K$$

Where ϕ is the uniform random number distributed between $-\pi$ to π The Signal-to-Noise Ratio (SNR) is defined as

$$SNR = 10 log_{10} \frac{\sigma_s^2}{\sigma_w^2}$$

where, σ_s^2 and σ_w^2 are signal power and noise power respectively. In the first example, the performance of the proposed IFKSA-ESPRIT algorithm is compared with the ESPRIT algorithm and Cramer Rao Lower Bound (CRLB) [14]. The Root Mean Square Error (RMSE) of the estimated DOAs are calculated and plotted with respect to various Signal-to-Noise Ratio (SNR). The RMSE is calculated using the equation,

$$RMSE = \sqrt{E\left\{\left(\theta - \hat{\theta}\right)^2\right\}}$$

From Fig. 4, it can be seen that for the case of ULA the IFKSA-ESPRIT is showing improved performance when compared with the ESPRIT algorithm and is close to CRLB at low SNR. In the next example, the performance of the proposed algorithm is evaluated for SLA. Various SLA geometries assumed for the simulation is listed in Table. IV. The numbers in the table gives the information on the position of the elements feeding the output and the missing numbers means the corresponding element is missing. The estimated DOAs are also plotted on histogram with respect to number of trails. The histogram plots give the information on the accuracy of the estimator to estimate the DOAs. From the RMSE vs. SNR plot for all the four cases is shown in Fig. 5, it can be observed that except for case 2, in all other cases the algorithm is able to estimate the DOAs. It can be observed from Table. IV, for second case the number of elements are very less, resulting in performance degradation. The histogram for all the three cases (See Fig. 6 - Fig. 9) show that for the second case of SLA the accuracy is poor, see Fig. 7.

TABLE IV. ELEMENTS CONSIDERED TO FORM SLA IN SIMULATION

ELEMENTS CONSIDERED TO FORM SLA IN SIMULATION

SLA	Positions
SLA1	[1,2,3,4,5,6,15,16,17,18,19,20]
SLA2	[1,2,3,4,5,14,17,20]
SLA3	[1,2,3,4,5,8,12,16,18,20]
SLA4	[1,2,3,5,6,7,10,12,13,14,16,17,19,20]

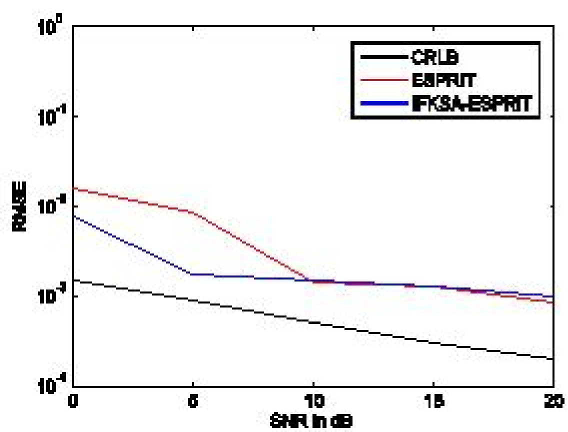

Figure 4: RMSE vs. SNR plot of IFKSA-ESPRIT, ESPRIT and CRLB

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Figure 9:Histogram of the estimated DOAs fro case 4. Assumed DOAs of two signals are $[10^\circ, 4$. SNR = 10 dB

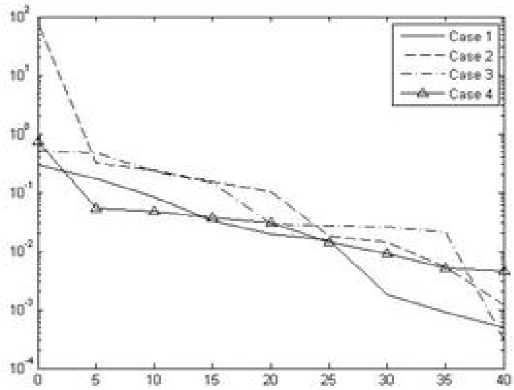

Figure. 5. RMSE vs. SNR plot of estimated DOAs for all the four cases of SLA. Assumed DOAs of two signals are $[10^\circ, 45^\circ]$

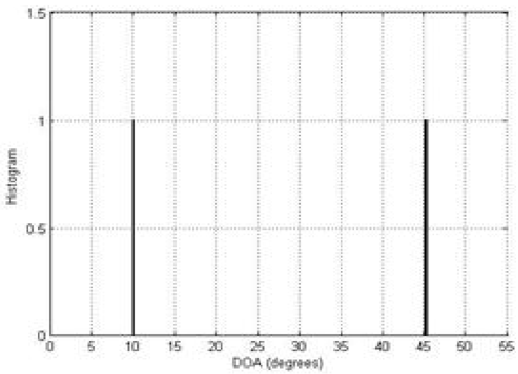

Figure. 6. Histogram of the estimated DOAs fro case 1. Assumed DOAs of two signals are $[10^\circ, 45^\circ]$ SNR = 10 dB

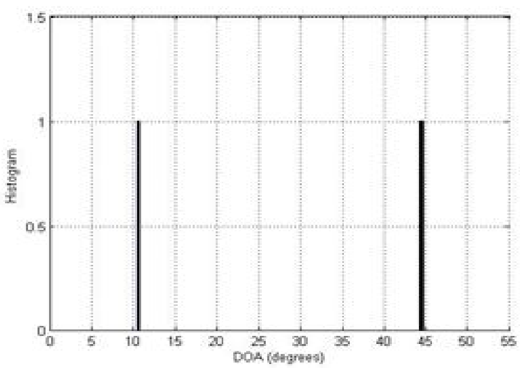

Figure. 7. Histogram of the estimated DOAs fro case 2. Assumed DOAs of two signals $[10^\circ, 45^\circ]$ SNR = 10 dB

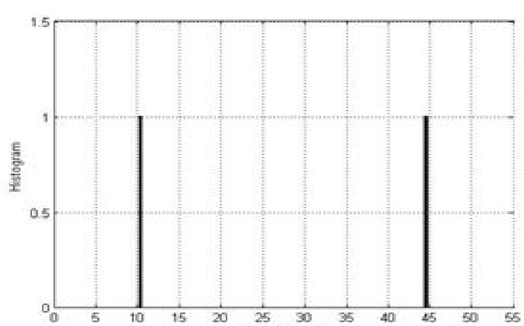

Figure. 8. Histogram of the estimated DOAs fro case 3. Assumed DOAs of two signals are $[10^\circ, 45^\circ]$ SNR = 10 dB.

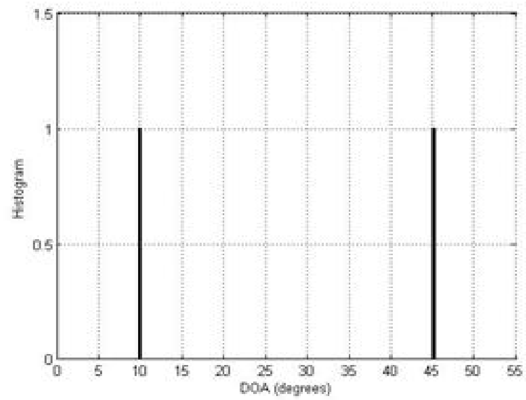

Figure. 9. Histogram of the estimated DOAs fro case 4. Assumed DOAs of two signals are $[10^\circ, 45^\circ]$ SNR = 10 dB

V. Conclusion

In this paper, we proposed IFKSA-ESPRIT DOA estimation algorithm which uses the Sparse Liner Array of antenna. The sparse data generated by SLA is first decomposed using IFKSA technique to obtain the signal subspace and later, the ESPRIT is applied to estimate the DOAs. Simulation results revealed that, for the ULA when compare to ESPRIT the proposed method performs better. The proposed method is evaluated for various SLA geometries and the algorithm succeeds in estimating the DOAs for all the cases. However, further investigation has to be made to handle the coherent signal sources.

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